Phase and Group Delay Estimation

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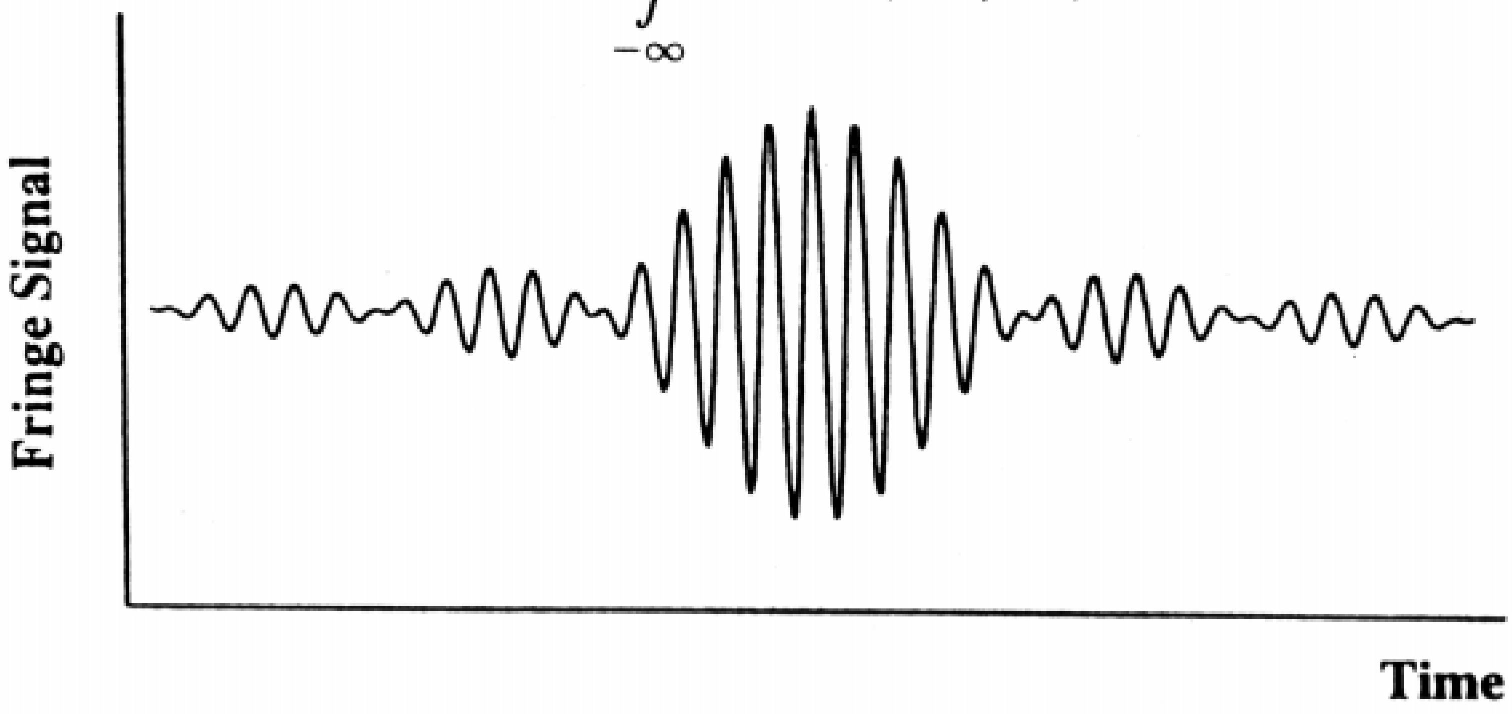
Who needs phase or group delay estimation?

- Acquisition of fringes
  - Observations must stay within the coherence envelope
  - By eye, ear, or by photoelectric means
- Calibration of fringe measurements
  - Must remain on the same part of the coherence envelope, otherwise calibration (source/calibrator) will not be accurate
- Astrometric interferometers measure phase
- Phase measurement necessary for image reconstruction

White light fringe

$$I_w(\kappa, x) = \int_{-\infty}^{\infty} W(\kappa)\, I(\kappa, x)\, d\kappa,$$



$$I(\kappa, x) = I_s \Big[1 + |\gamma_{12}| \cos(2\pi\kappa x - \phi_{12}) \Big] + I_b.$$

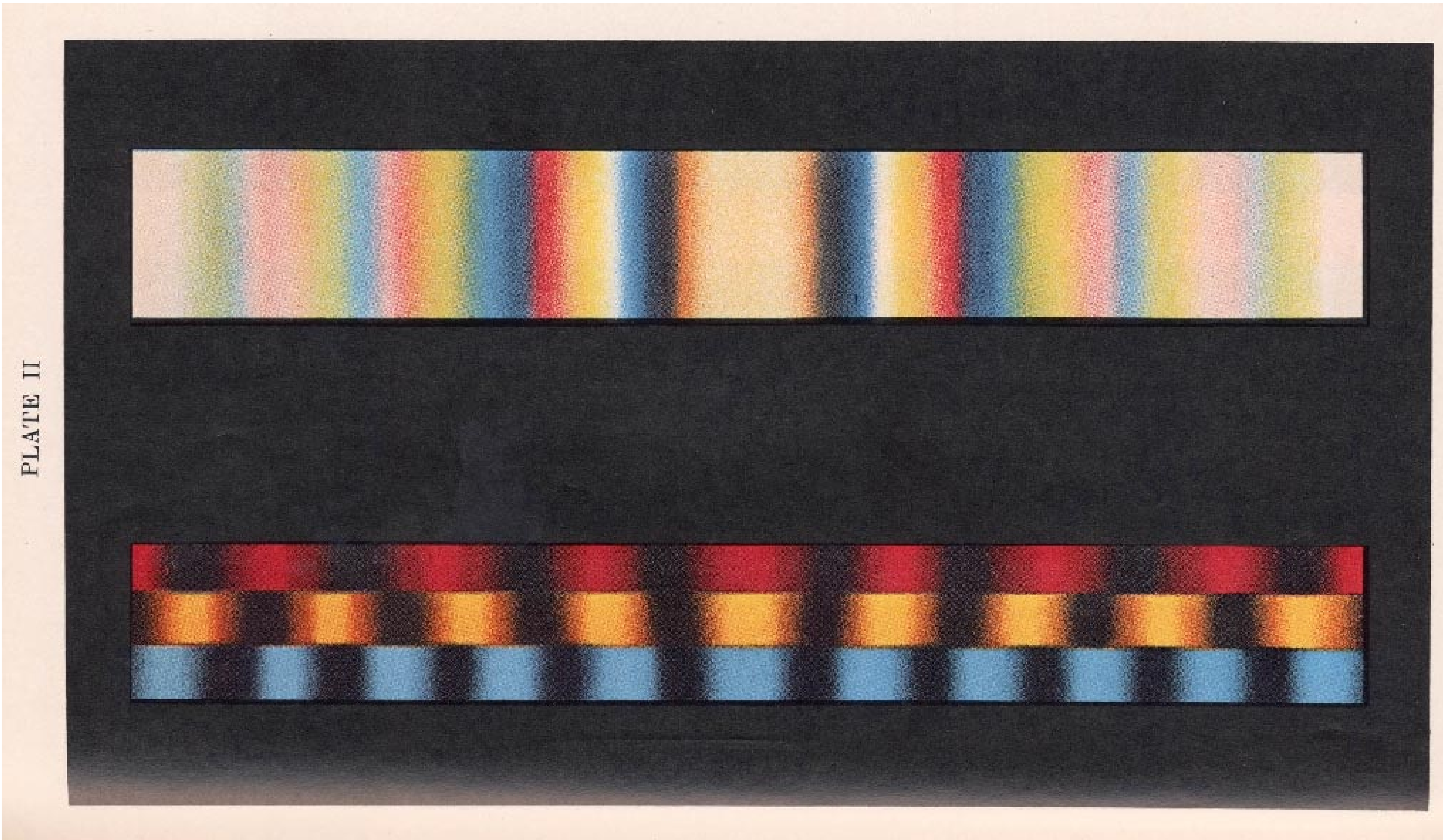


PLATE II

MEASUREMENT OF THE DIAMETER OF α ORIONIS WITH THE INTERFEROMETER[1]

BY A. A. MICHELSON AND F. G. PEASE

One of the wedges (*H*, Fig. 3), whose angles are about 10°, can be moved 25 mm either side of its mean position, parallel to the inclined surfaces. One turn of the rod (*J*, Plate IV*b*) shifts this wedge 0.5 mm, thus introducing an equivalent air path of about 0.045 mm. Although fringes can be observed throughout one-third of a turn, corresponding to an air path of 0.015 mm, or about 26 light-waves, the finding of the fringes is notably facilitated by a direct-vision prism (*K*, Plate IV*b*) placed in front of the eyepiece, which permits observation of interference bands with a path-difference of several hundred waves.

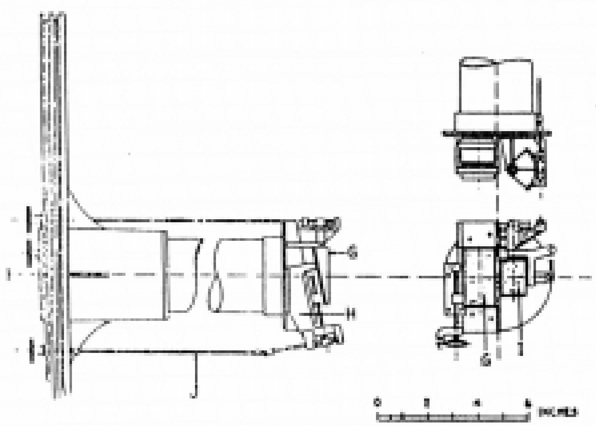


FIG. 3.—Diagram of adapter at focus. *G*, fixed wedge; *H*, movable wedge; *I*, plane-parallel compensator; *J*, rod to shift wedge.

INTERFERENCE FRINGES OBTAINED ON VEGA WITH TWO OPTICAL TELESCOPES

ANTOINE LABEYRIE

A slot and direct-view prism may be inserted in the output beam for observing the fringes in dispersed light. As experienced by Michelson, this facilitates considerably the fringe acquisition. Indeed, fringes are still visible in the spectrum with over 50 μ of path difference, and can thus usually be found within minutes since the position uncertainties are in the order of a millimeter. Once acquired in dispersed light, the fringes are easily found in white light. Because the autoguider system which drives both telescopes may be replaced by human guiders, low-cost versions of such interferometers could probably be constructed by experienced amateurs.

MICHELSON STELLAR INTERFEROMETRY

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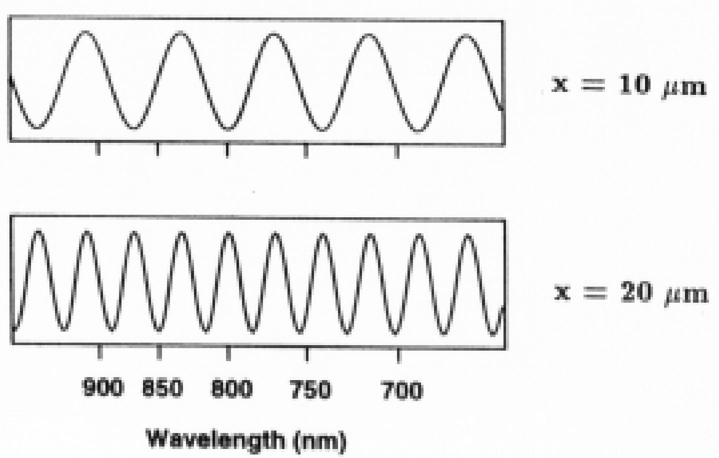
(2) *Automatic Fringe Tracking* (*AFT*). There is a very real possibility that AFT can be used in the near future. Simple tracking for a short baseline has already been proposed by an "astrometric" interferometer (SHAO and STAELIN [1977]), and HARDY and WALLINER [1979] have examined the problem from the point of view of the active control system. It has been pointed out (STEEL [1978]) that if a multiple spectral band detector is used a particularly simple AFT can be based on the fact that the spectrum is "channeled" with fringes, the period of which is inversely proportional to the path error. With AFT considerably longer baselines should be possible. The performance of an AFT system will be limited by the available signal and the integration time of the servo system. As the variation in path is quite slow, very small bandwidths can be used. The signal will depend both on the brightness of the source and the fringe visibility. For very long baselines, of course, many objects will be partially resolved and AFT may be less effective, but it seems likely that if it can be used the chief limitation to resolution will be the availability of suitably large sites for the interferometer.

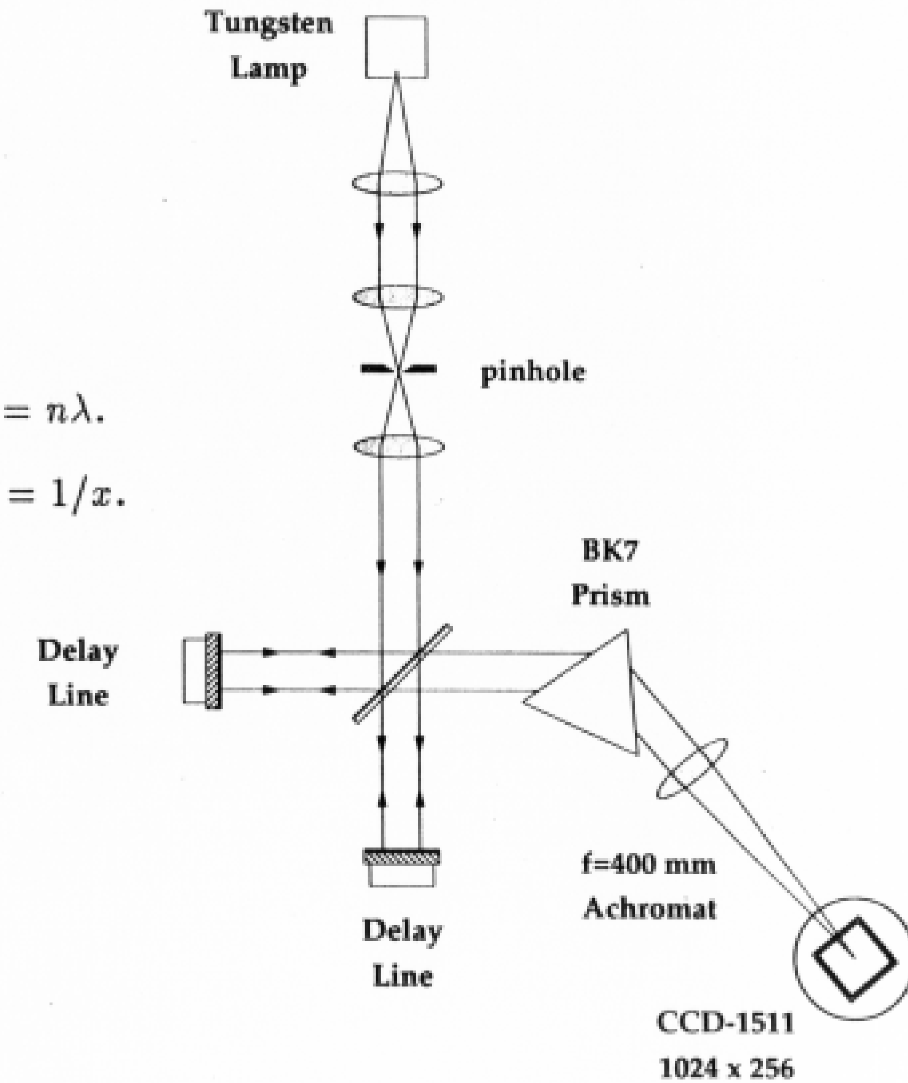
Channeled Spectrum

Fringes in dispersed light.

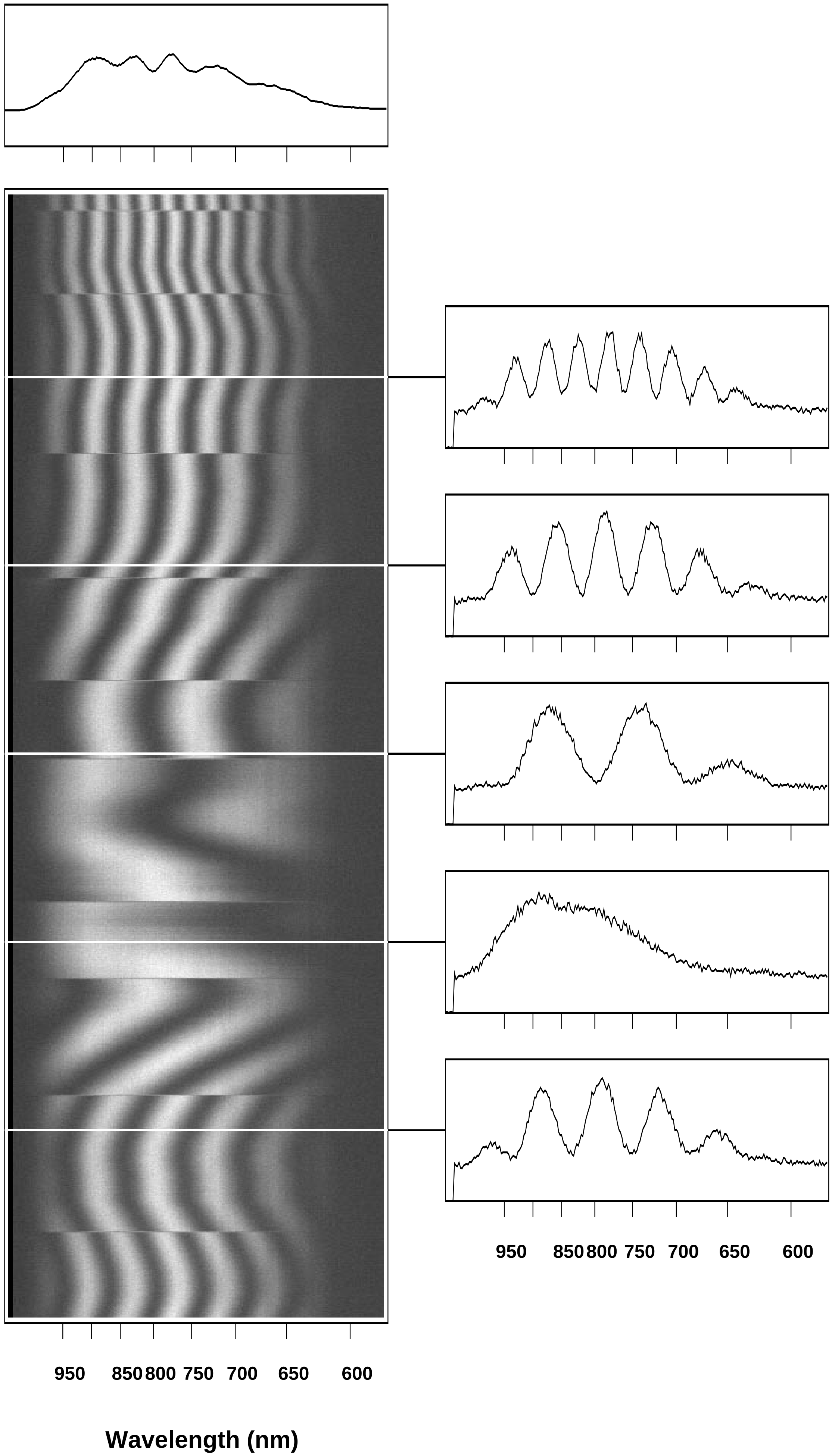
$$I(\lambda) = I_s \left[1 + |\gamma| \cos\left(\frac{2\pi x}{\lambda} - \alpha \right) \right] + I_b.$$

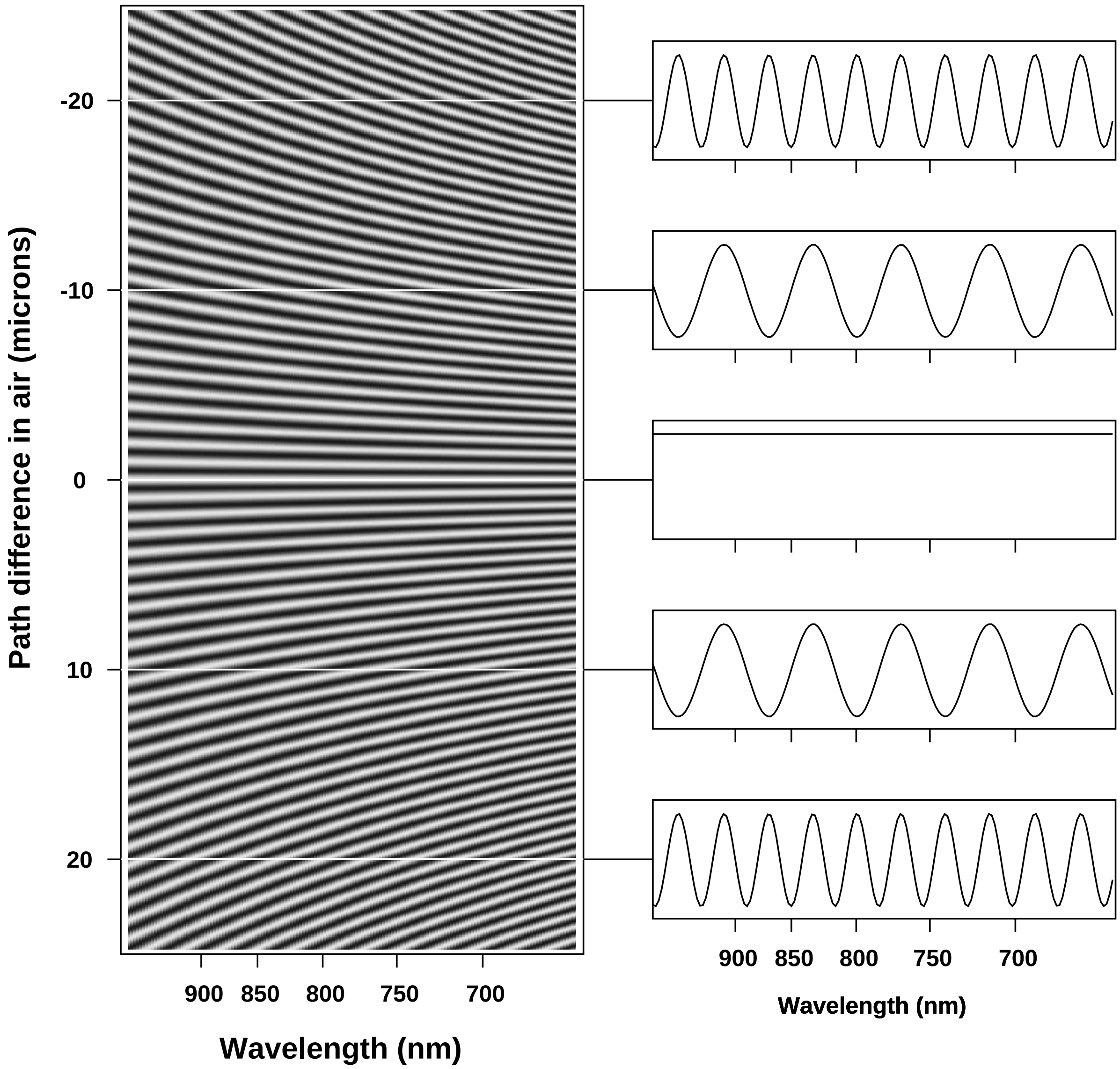
- Bright fringes in spectrum whenever $x = n\lambda$.
- Fringe period in wavenumber is $\Delta(1/\lambda) = 1/x$.





/coast/lawson/june3/jun_9




-20
-10
0
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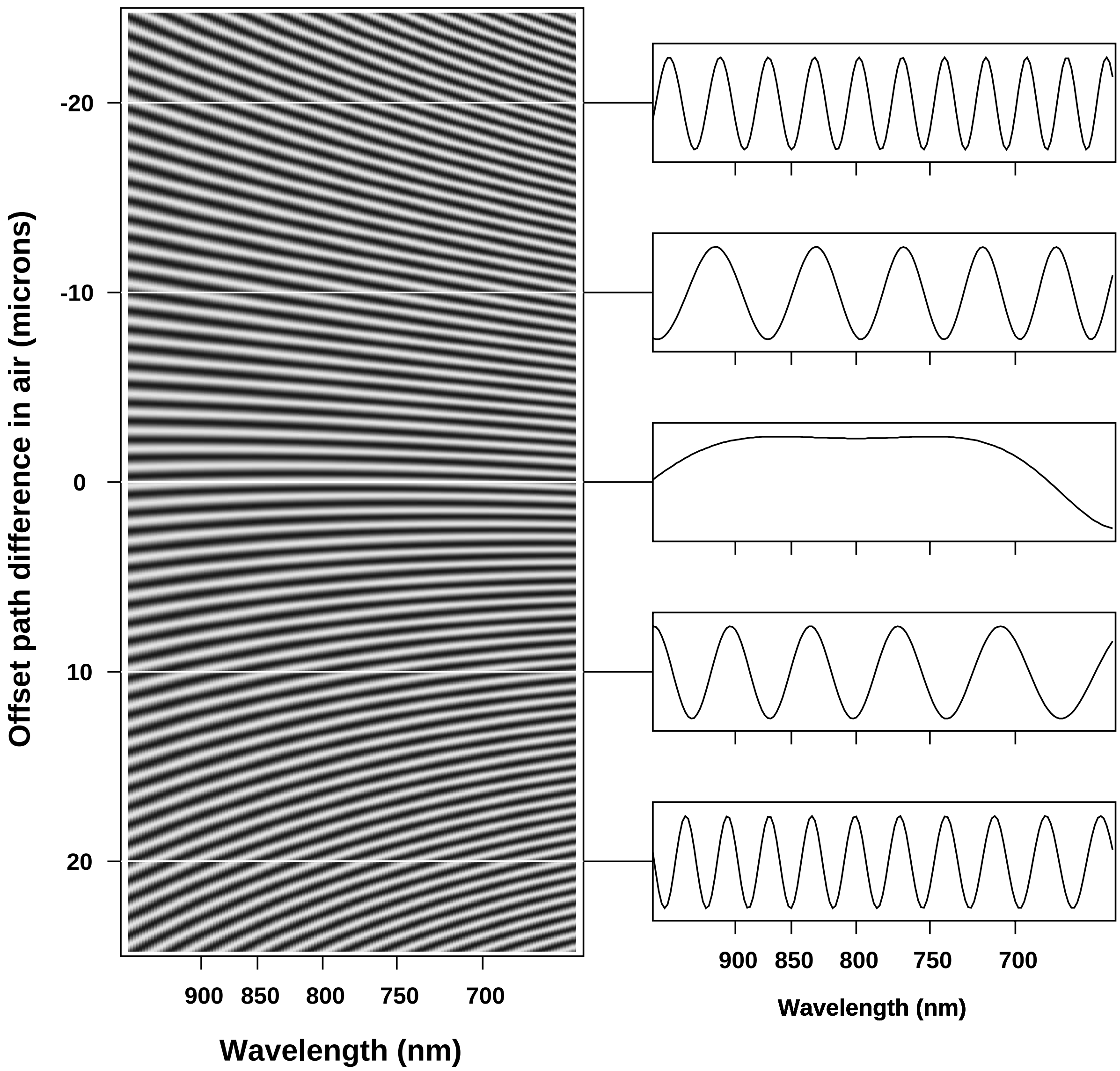
Path difference in air (microns)
900 850 800 750 700
Wavelength (nm)
900 850 800 750 700
Wavelength (nm)



Offset path difference in air (microns)
-20
-10
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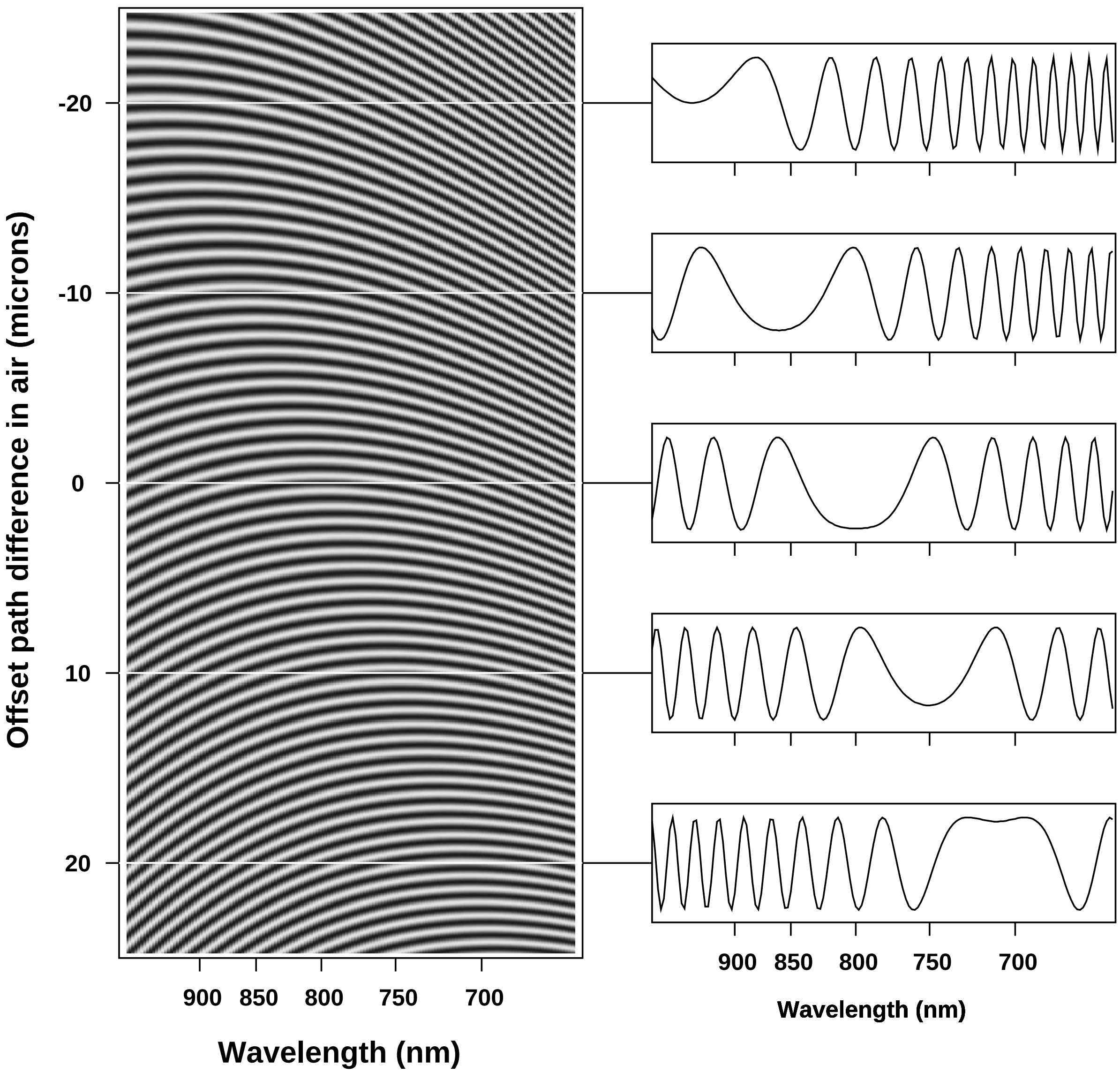
900 850 800 750 700
Wavelength (nm)
900 850 800 750 700
Wavelength (nm)


Offset path difference in air (microns)
-20
-10
0
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Wavelength (nm)
900
850
800
750
700
Wavelength (nm)

Phase Delay and Group Delay Explained

- **Phase Delay: phase difference at a particular wavenumber**

$$\begin{aligned} x(\kappa) &= x_0 + x_1 n_1(\kappa) \\ 2\pi\kappa x(\kappa) &= 2\pi\kappa\,[x_0 + x_1 n_1(\kappa)] \end{aligned}$$

- **Group Delay: Rate-of-change of phase w.r.t. wavenumber measured at a particular wavenumber (1/λ).**

$$\begin{aligned} \mathrm{GD}(\kappa_0) &= \frac{1}{2\pi}\,\frac{d}{d\kappa}\,[2\pi\kappa\,\{x_0 + x_1 n_1(\kappa)\}]\Big|_{\kappa_0} \\ \mathrm{GD}(\kappa_0) &= x_0 + x_1\,\frac{d}{d\kappa}\,[\kappa n_1(\kappa)]\Big|_{\kappa_0} \end{aligned}$$

Model of the Fringe

$$I(\kappa) = I_s\left[1 + |\gamma_x| \cos\left(2\pi\kappa x - \phi_{12} + \phi_0\right)\right] + I_b,$$

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b),$$

$$I(\kappa) = C_0 + X\cos(2\pi\kappa x) + Y\sin(2\pi\kappa x),$$

$$\begin{aligned} C_0 &= I_s \\ X &= I_s\,|\gamma_{12}|\,\cos(\phi_{12} - \phi_0) \\ Y &= I_s\,|\gamma_{12}|\,\sin(\phi_{12} - \phi_0) \end{aligned}$$

Methods of Phase Estimation: phase stepping technique

$$I(\kappa, n) = C_0 + X\cos(2\pi\kappa x_n) + Y\sin(2\pi\kappa x_n),$$

$$x_n = \frac{n}{4}\frac{1}{\kappa_0}, \qquad n = 0, 1, 2, 3$$

$$\begin{aligned} A &= C_0 + X \\ B &= C_0 + Y \\ C &= C_0 - X \\ D &= C_0 - Y \end{aligned}$$

$$\begin{aligned} A + B + C + D &= 4C_0, \\ A - C &= 2X, \\ B - D &= 2Y. \end{aligned}$$

$$\phi_{12} - \phi_0 = \tan^{-1}\left(\frac{B - D}{A - C}\right),$$

$$|\gamma_{12}| = \frac{1}{2}\,\frac{\sqrt{(A-C)^2 + (B-D)^2}}{(A+B+C+D)/4}.$$

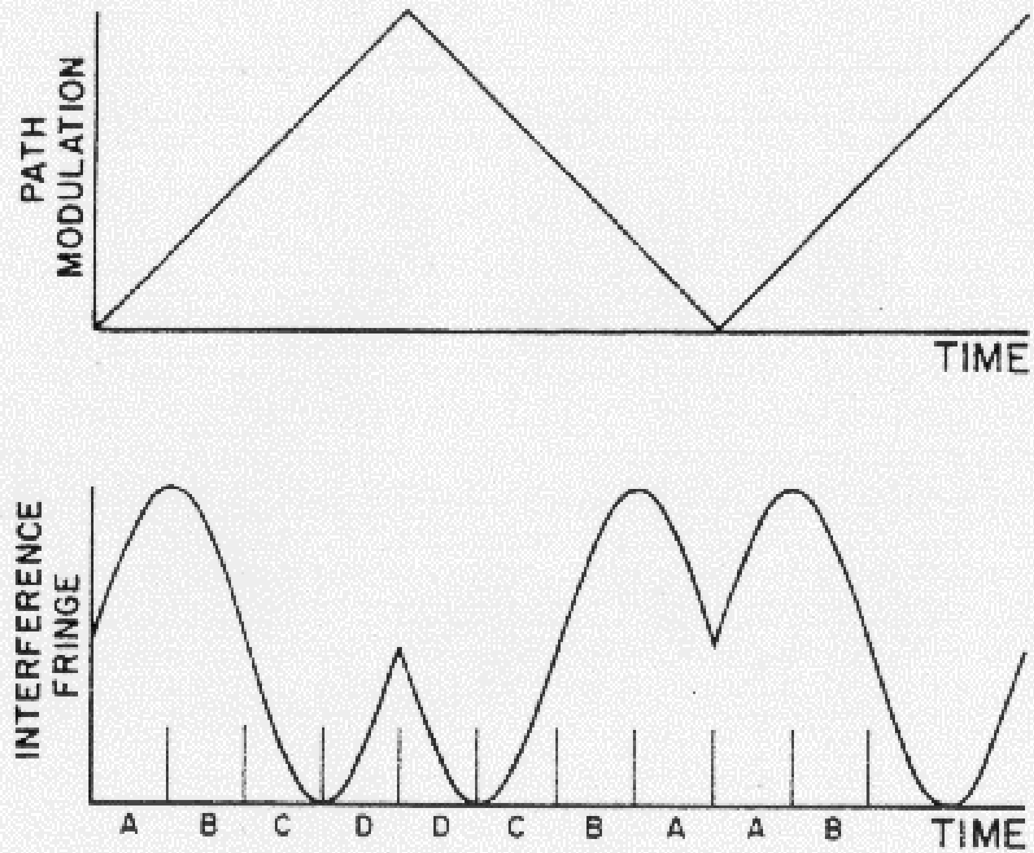


Fig. 6. Stellar fringe modulation/demodulation

Methods of Phase Estimation: integrating bucket technique

$$\int_{x_n}^{x_{n+1}} I(\kappa, n)\, dx.$$

$$\int_{x_n}^{x_{n+1}} \cos(2\pi\kappa x)\, dx = \frac{1}{2\pi\kappa}\left[\sin(2\pi\kappa x_{n+1}) - \sin(2\pi\kappa x_n)\right],$$

$$\int_{x_n}^{x_{n+1}} \sin(2\pi\kappa x)\, dx = \frac{1}{2\pi\kappa}\left[\cos(2\pi\kappa x_n) - \cos(2\pi\kappa x_{n+1})\right].$$

$$x_n = \frac{n}{4}\frac{1}{\kappa_0} - \frac{3}{4}\frac{1}{\kappa_0}, \qquad n = 1, 2, 3, 4, 5$$

$$
\begin{aligned}
A &= \frac{C_0}{4\kappa_0} + \frac{X}{2\pi\kappa_0}\left[\sin(-\pi/2) - \sin(-\pi)\right] + \frac{Y}{2\pi\kappa_0}\left[\cos(-\pi) - \cos(-\pi/2)\right] \\
B &= \frac{C_0}{4\kappa_0} + \frac{X}{2\pi\kappa_0}\left[\sin(0) - \sin(-\pi/2)\right] + \frac{Y}{2\pi\kappa_0}\left[\cos(-\pi/2) - \cos(0)\right] \\
C &= \frac{C_0}{4\kappa_0} + \frac{X}{2\pi\kappa_0}\left[\sin(\pi/2) - \sin(0)\right] + \frac{Y}{2\pi\kappa_0}\left[\cos(0) - \cos(\pi/2)\right] \\
D &= \frac{C_0}{4\kappa_0} + \frac{X}{2\pi\kappa_0}\left[\sin(\pi) - \sin(\pi/2)\right] + \frac{Y}{2\pi\kappa_0}\left[\cos(\pi/2) - \cos(\pi)\right]
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{C_0}{4\kappa_0} - \frac{X}{2\pi\kappa_0} - \frac{Y}{2\pi\kappa_0} &&= \frac{1}{2\pi\kappa_0}\left[\frac{\pi}{2}C_0 - X - Y\right] \\
B &= \frac{C_0}{4\kappa_0} + \frac{X}{2\pi\kappa_0} - \frac{Y}{2\pi\kappa_0} &&= \frac{1}{2\pi\kappa_0}\left[\frac{\pi}{2}C_0 + X - Y\right] \\
C &= \frac{C_0}{4\kappa_0} + \frac{X}{2\pi\kappa_0} + \frac{Y}{2\pi\kappa_0} &&= \frac{1}{2\pi\kappa_0}\left[\frac{\pi}{2}C_0 + X + Y\right] \\
D &= \frac{C_0}{4\kappa_0} - \frac{X}{2\pi\kappa_0} + \frac{Y}{2\pi\kappa_0} &&= \frac{1}{2\pi\kappa_0}\left[\frac{\pi}{2}C_0 - X + Y\right]
\end{aligned}
$$

We can now write

$$\frac{B-D}{A-C}=\frac{Y-X}{X+Y}.$$

Dividing through by X and using the relationship $Y/X=\tan(\phi_{12}-\phi_0)$,

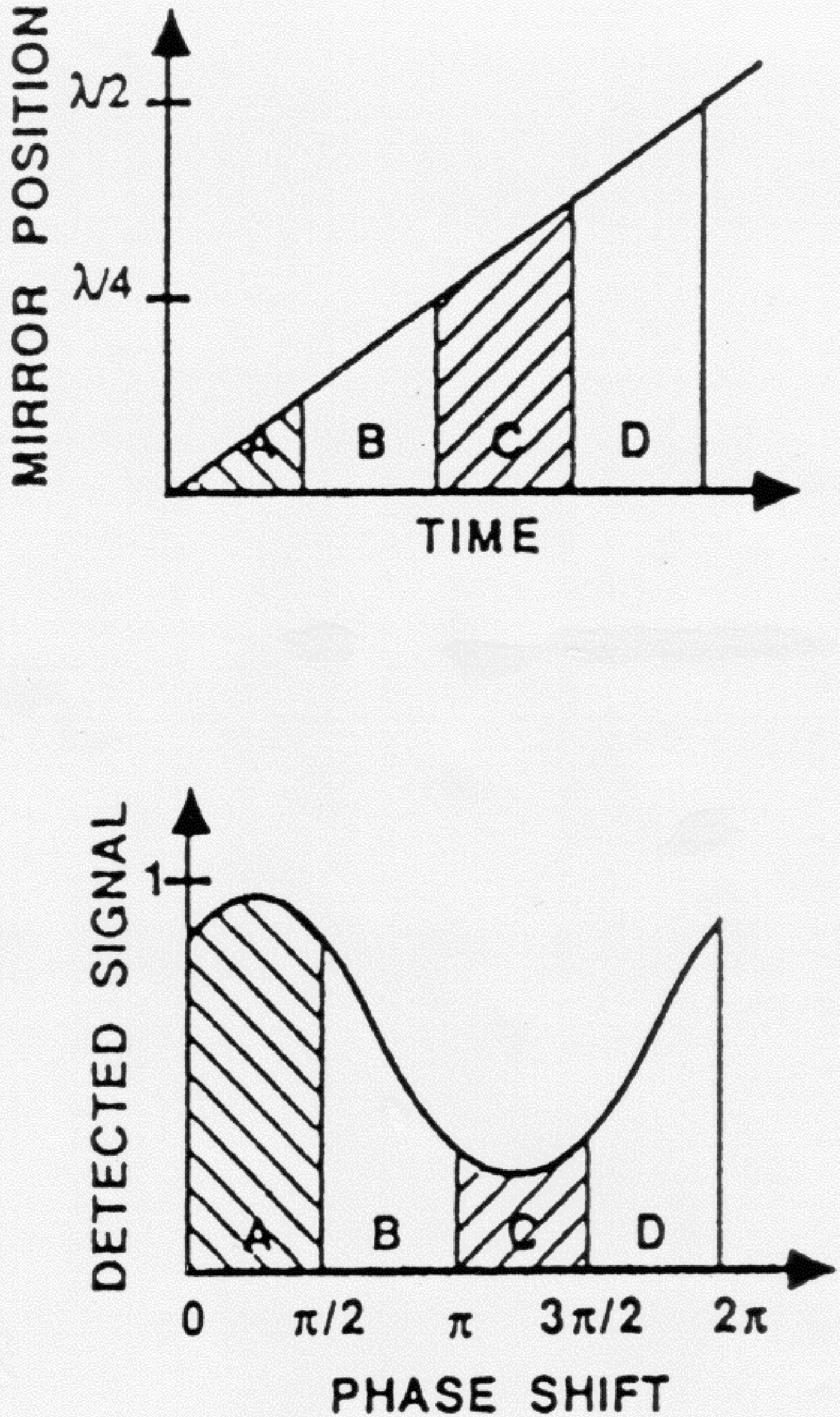
$$\frac{B-D}{A-C}=\frac{\tan(\phi_{12}-\phi_0)-1}{1+\tan(\phi_{12}-\phi_0)}.$$

Using the trigonometric relationship

$$\tan(a-b)=\frac{\tan a-\tan b}{1+\tan a\tan b},$$

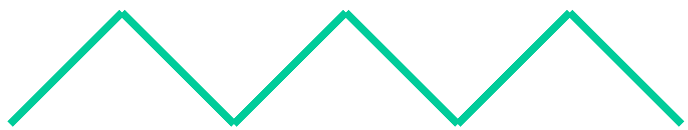
we have finally that

$$\begin{aligned}\tan^{-1}\left(\frac{B-D}{A-C}\right) &= \phi_{12}-\phi_0-\frac{\pi}{4},\\ \phi_{12}-\phi_0 &= \tan^{-1}\left(\frac{B-D}{A-C}\right)+\frac{\pi}{4}.\end{aligned}$$

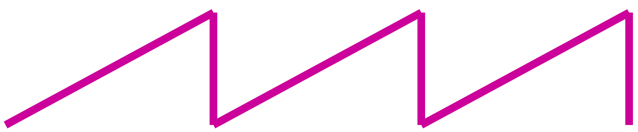

MIRROR POSITION
λ/2
λ/4
A
B
C
D
TIME
DETECTED SIGNAL
1
A
B
C
D
0
π/2
π
3π/2
2π
PHASE SHIFT

Methods of Phase Measurement
integrating bucket technique
Sources of Error

- **Vibration and turbulence**
- **Quantization of detected signal**
- **Miscalibration of phase-shifter (piezo stroke)**
- **Phase-shifter stroke non-linearities**
  - **require same non-linearities for each measurement**



Triangular waveform (Mark III, NPOI)



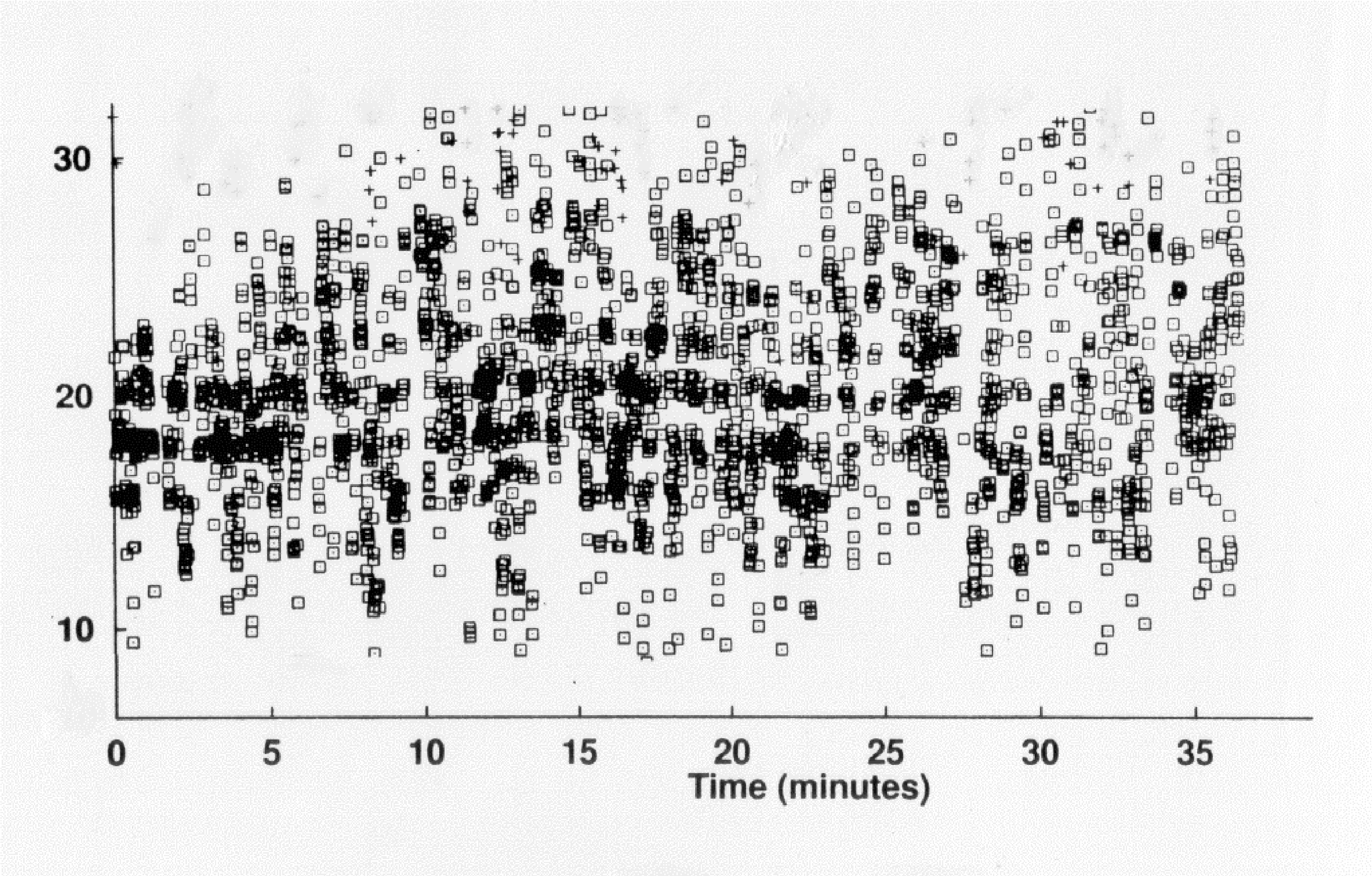
Sawtooth waveform (PTI)

- **Detector non-linearities**

Methods of Phase Estimation: simultaneous phase measurements at multiple wavelengths

- **Mark III Interferometer**
  - **Delay modulated with a stroke of λ=800 nm**
  - **Observations at 550 nm and 450 nm re-define the bounds of the A, B, C, and D bins because the total stroke is greater than 1λ at those wavelengths**
  - **Photons are thrown away when stroke > λ**
- **PTI and NPOI use this principle for multi-wavelength phase measurements**
  - **PTI uses NICMOS III detectors at 2.2 μm**
  - **NPOI uses avalanche photodiodes at visible wavelengths**

Phase Unwrapping



Group Delay Tracking

- **Number of fringes across bandwidth defined by λ_{min} and λ_{max} is given by**

$$p = \frac{1}{2\pi}\left[\frac{2\pi x}{\lambda_{\mathrm{min}}} - \frac{2\pi x}{\lambda_{\mathrm{max}}}\right],$$

- **Path-difference is proportional to the number of fringes**

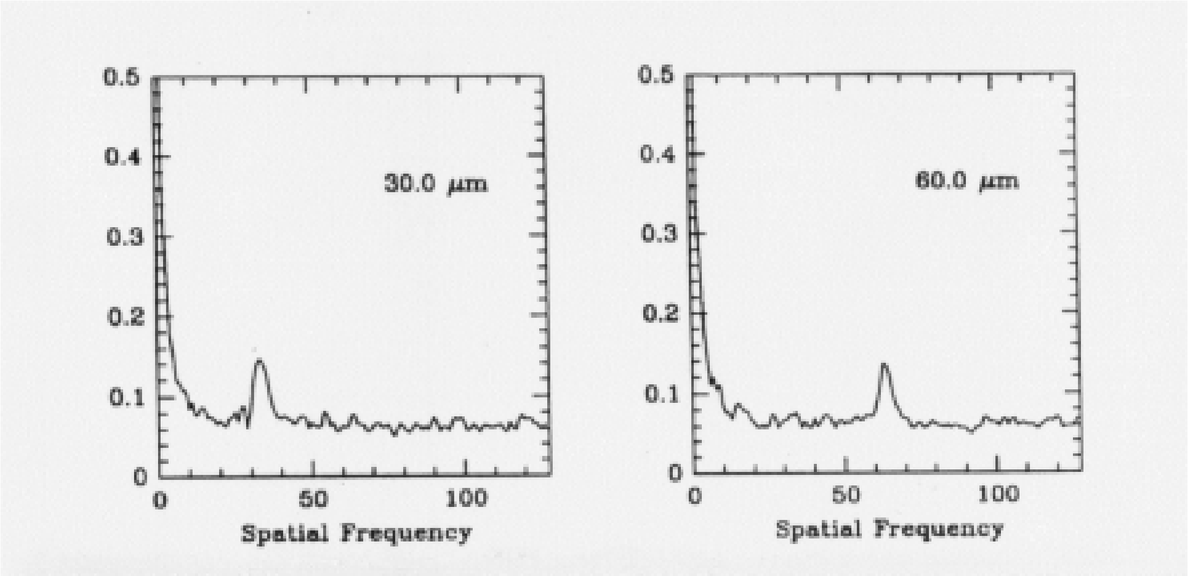
$$x = \frac{p}{\Delta\kappa},$$

$$\Delta\kappa = \left[\frac{1}{\lambda_{\mathrm{min}}} - \frac{1}{\lambda_{\mathrm{max}}}\right].$$

Approaches to Group Delay Tracking

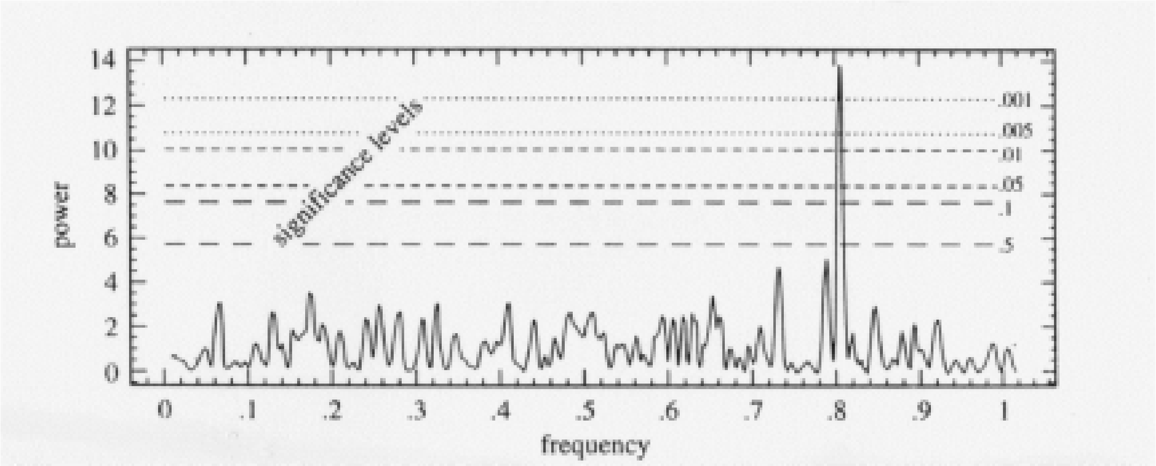
- **Data**
  1. **Channeled spectrum**: single snap-shot of stellar spectrum with fringes.
  2. **Fringe phasors**: derived from multi-wavelength phase measurements.
- **Data processing algorithms**
  1. **Fast Fourier transform**, or any of its relatives.
  2. **Least-squares fit** using model stellar fringes or phasors.
  3. Any (model-based) method of modern spectrum analysis.

Methods of Group Delay Estimation:
Channeled Spectrum
fast Fourier transform



- **Disadvantages:**
  - **No sign (+/-) for group delay**
  - **Tracking at zero group delay is problematic**
  - **Thresholding not straightforward**

Methods of Group Delay Estimation:
Channeled Spectrum
Lomb-Scargle Periodogram



- **Advantages:**
  - **Linear least-squares fit to harmonics**
  - **Allows threshold to be set for peak detection**
- **Disadvantages:**
  - **Not easily adaptable to fringe phasors ?**

Other methods of least-squares fitting also possible using more parameters
See for example W.A. Traub *et al*. SPIE **1237**, 145-152 (1990)

Methods of Group Delay Estimation:
Multi-Wavelength Fringe Phasors
fast Fourier transform

If we measure the quantities A, B, C, D, at M wavenumbers we can calculate

$$\begin{aligned} h_c(\kappa_m) &= A(\kappa_m) - C(\kappa_m) \qquad m = 0, ..., M-1 \\ h_s(\kappa_m) &= B(\kappa_m) - D(\kappa_m) \qquad m = 0, ..., M-1 \end{aligned}$$

so that we now have

$$\begin{aligned} h_c(\kappa_m) &\propto \cos(2\pi\kappa_m x), \\ h_s(\kappa_m) &\propto \sin(2\pi\kappa_m x). \end{aligned}$$

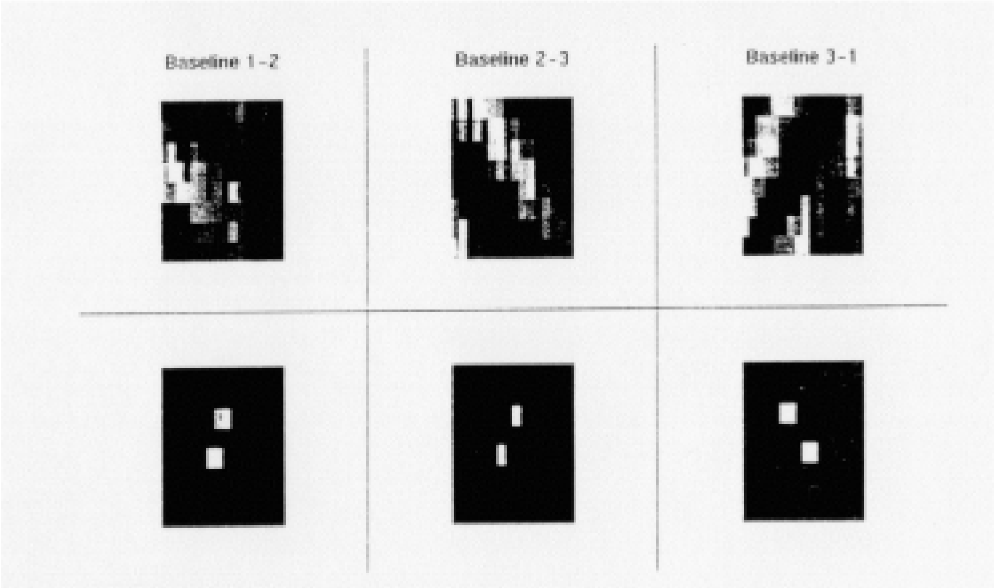
We can now define the complex number series

$$h(\kappa_m) = h_c(\kappa_m) + j h_s(\kappa_m), \qquad m = 0, ..., M-1$$

and take its discrete Fourier transform

$$H(x) = \sum_{m=0}^{M-1} h(\kappa_m) \exp(j2\pi\kappa_m x).$$

Methods of Group Delay Estimation:
Multi-Wavelength Fringe Phasors
fast Fourier transform



- **Advantages:**
  - **sign is unambiguous**
  - **tracking at zero group delay is straightforward**
- **Disadvantages:**
  - **Cannot weigh data to selectively ignore bad pixels**

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